NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1346

CRITICAL SHEAR STRESS OF LONG PLATES WITH TRANSVERSE CURVATURE

By S. B. Batdorf, Murry Schildcrout, and Manuel Stein

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

Washington
June 1947

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1346

CRITICAL SHEAR STRESS OF LONG PLATES WITH TRANSVERSE CURVATURE

By S. B. Batdorf, Murry Schildcrout, and Manuel Stein

SUMMARY

A theoretical solution is presented for the buckling stress of long plates with transverse curvature loaded in shear. The results are given in the form of curves and formulas for plates with simply supported or clamped edges for a wide range of plate dimensions. Comparisons are made with theoretical solutions obtained in previous investigations.

INTRODUCTION

The problem of determining the buckling stress of long plates with transverse curvature subjected to shear has received limited treatment in two previous investigations. In a paper published in 1937 (reference 1), Leggett presented solutions for plates with simply supported edges and for plates with clamped edges. Numerical results, however, could be given only for plates of small curvature. In 1938 Kromm (reference 2) published a solution which covered the complete curvature range but considered only simple support. The present paper treats plates with both simply supported and clamped edges over the complete curvature range. The boundary conditions on median-surface displacement in Leggett's analysis differ from those in Kromm's analysis and in the present work.

The various boundary conditions treated are summarized in the following table:

Authors	Boundary conditions				
	Displacement			Slope	Moment
	Axial	Circum-ferential	Radial		
	Edges simply supported				
Leggett	0	0	0	Unrestrained	0
Kromm	0	Unrestrained	0	Unrestrained	0
Present paper	0	Unrestrained	0	Unrestrained	0
	Edges clamped				
Leggett	0	0	0	0	Unrestrained
Present paper	Unrestrained	0	0	0	Unrestrained
	0	Unrestrained	0	0	Unrestrained

These various assumed boundary conditions lead to different values for the critical stresses. The results found for the sets of boundary conditions given in the preceding table are discussed and compared in the following section.

RESULTS AND DISCUSSION

The critical shear stress for a long plate with transverse curvature is given by the equation

$$\tau_{cr} = \frac{k_s \pi^2 D}{b^2 t}$$

where

b width of plate measured along arc

t thickness of plate

D flexural stiffness of plate per unit length $\left(\frac{Et^3}{12\left(1 - \mu^2\right)}\right)$

E Young's modulus of elasticity

k_s critical-shear-stress coefficient

τ_{cr} critical shear stress

μ Poisson's ratio

The critical-shear-stress coefficients k_s found by the methods of the present paper (see appendix) for plates with both simply supported and clamped edges are given in tables 1 and 2 and are plotted in figure 1. The ordinate in this figure is the critical-shear-stress coefficient, and the abscissa is a curvature parameter Z which depends on the dimensions of the plate and on Poisson's ratio as follows:

$$Z = \frac{b^2}{rt}\sqrt{1 - \mu^2} = \left(\frac{b}{r}\right)^2 \frac{r}{t}\sqrt{1 - \mu^2}$$

where r is the radius of curvature of the plate.

As the curvature parameter Z becomes small and approaches zero, the value of the shear-stress coefficient k_s for simply supported edges approaches the known value for a flat plate of 5.34. The two solutions given for plates with clamped edges correspond to different boundary conditions on median-surface displacement, as indicated in figure 1. In the solution represented by the solid curve the value of k_s approaches the established value for a flat plate ($k_s = 8.98$) as Z approaches zero. In the solution represented by the dashed curve the value of k_s as Z approaches zero is about 7 percent higher because of poor convergence. The solution leading to the solid curve was so rapidly convergent that fourth-order determinants were found satisfactory. Tenth-order determinants were used to obtain the dashed curve, and the additional labor required to obtain an accuracy comparable to that of the solid curve was considered prohibitive.

As Z increases k_s also increases and the curves approach the straight lines given by the following equations. For plates with simply supported edges,

$$k_s = 1.9Z^{1/2}$$

For plates with clamped edges (v unrestrained; u = 0 at edges),

$$k_s = 3.1Z^{1/2}$$ (Result probably somewhat high because of poor convergence)

For plates with clamped edges (v = 0; u unrestrained at edges),

$$k_s = 3.4Z^{1/2}$$

These equations apply when $Z > 20$.

In figure 2 the results of the present paper are compared with those given by Leggett (reference 1). Leggett's results for the critical stresses of plates with simply supported edges are considerably higher than those of the present paper (which presents results identical with those of Kromm) because of the additional restraint imposed upon the plate in Leggett's solution. For plates with clamped edges all solutions give approximately the same results in the low curvature range to which Leggett's results are restricted.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., March 20, 1947

APPENDIX

Symbols

b width of plate measured along arc

m,n,j integers

r radius of curvature of plate

t thickness of plate

u displacement of point on median surface of plate in axial (x-) direction

v displacement of point on median surface of plate in circumferential (y-) direction

w displacement of point on median surface of plate in radial direction, positive outward

x axial coordinate of plate

y circumferential coordinate of plate

D flexural stiffness of plate per unit length $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$

E Young's modulus of elasticity

Q mathematical operator

Z curvature parameter $\left(\frac{b^2}{rt}\sqrt{1 - \mu^2} \text{ or } \left(\frac{b}{r}\right)^2 \frac{r}{t}\sqrt{1 - \mu^2}\right)$

a_n, b_n coefficients of deflection functions

k_s critical-shear-stress coefficient appearing in formula

$$\tau_{cr} = \frac{k_s \pi^2 D}{b^2 t}$$

$$M_n = \frac{\pi}{8\beta}\left[\left(n^2 + \beta^2\right)^2 + \frac{12Z^2\beta^4}{\pi^4\left(n^2 + \beta^2\right)^2}\right]$$

V_m, W_m deflection functions

$\beta = \frac{b}{\lambda}$

λ half wave length of buckles in axial direction

μ Poisson's ratio

τ_{cr} critical shear stress

$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$

∇^{-4} = inverse of ∇^4, defined by $\nabla^{-4}\left(\nabla^4 w\right) = \nabla^4\left(\nabla^{-4} w\right) = w$

Theoretical Solution

Equation of equilibrium. - The critical shear stress at which buckling of a long plate with transverse curvature occurs may be obtained by solving the following equation of equilibrium (reference 3):

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + 2\tau_{cr}\, t\,\frac{\partial^2 w}{\partial x\,\partial y} = 0 \qquad (1)$$

Division of equation (1) by D gives the equation

$$\nabla^4 w + \frac{12Z^2}{b^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + 2k_s\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x\,\partial y} = 0 \qquad (2)$$

where the dimensionless parameters Z and k_s are defined by

$$Z = \frac{b^2}{rt}\sqrt{1 - \mu^2}$$

and

$$k_s = \frac{\tau_{cr} t b^2}{D\pi^2}$$

Equation (2) can be represented by

$$Qw = 0 \tag{3}$$

where Q is defined as the operator

$$\nabla^4 + \frac{12Z^2}{b^4} \nabla^{-4} \frac{\partial^4}{\partial x^4} + 2k_s \frac{\pi^2}{b^2} \frac{\partial^2}{\partial x\, \partial y}$$

<u>Method of solution</u>.- Equation (3) may be solved by the Galerkin method, which is outlined in reference 4. In the application of this method the displacement w is expressed in terms of a suitable series expansion as follows:

$$w = \sum_{m=1}^{j} a_m V_m + \sum_{m=1}^{j} b_m W_m \tag{4}$$

In equation (4) each of the functions V_1, V_2, ... V_j, W_1, W_2, ... W_j must satisfy the boundary conditions on w but need not satisfy the equation of equilibrium. The coefficients a_m and b_m are determined by the equations

$$\left.\begin{aligned} \int_0^{2\lambda} \int_0^{b} V_n Q_w \, dx\, dy = 0 \\ \int_0^{2\lambda} \int_0^{b} W_n Q_w \, dx\, dy = 0 \end{aligned}\right\} \tag{5}$$

where $n = 1, 2, 3, \ldots$

The boundary conditions considered in the present paper are as follows: for plates with simply-supported edges,

$w = \dfrac{\partial^2 w}{\partial y^2} = u = 0$ and v is unrestrained; for plates with clamped edges, $w = \frac{\partial w}{\partial y} = v = 0$ and u is unrestrained (case I) and $w = \frac{\partial w}{\partial y} = u = 0$ and v is unrestrained (case II).

Solution for plates with simply supported edges.- The following infinite series expansion represents the displacement w of curved plates with simply supported edges:

$$w = \sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi y}{b} + \cos\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} b_m \sin\frac{m\pi y}{b} \tag{6}$$

where λ is the half wave length of the buckles in the axial direction. Equation (6) satisfies the conditions on w for simple support and when introduced into equation (5) implies that the axial displacement u is equal to 0 and the circumferential displacement v is unrestrained at the edges (see reference 3). Equation (6) is equivalent to equation (4) if

$$\left.\begin{aligned} V_n &= \sin\frac{\pi x}{\lambda}\sin\frac{n\pi y}{b} \\ W_n &= \cos\frac{\pi x}{\lambda}\sin\frac{n\pi y}{b} \end{aligned}\right\} \tag{7}$$

where $n = 1, 2, 3, \ldots$

Substitution of expressions (6) and (7) into equations (5) and integration between the limits indicated give

$$\left.\begin{aligned} a_n\left[\left(n^2 + \beta^2\right)^2 + \frac{12}{\pi^4}\frac{\beta^4 Z^2}{\left(n^2 + \beta^2\right)^2}\right] - \frac{8\beta k_s}{\pi}\sum_{m=1}^{\infty} b_m \frac{mn}{n^2 - m^2} &= 0 \\ b_n\left[\left(n^2 + \beta^2\right)^2 + \frac{12}{\pi^4}\frac{\beta^4 Z^2}{\left(n^2 + \beta^2\right)^2}\right] + \frac{8\beta k_s}{\pi}\sum_{m=1}^{\infty} a_m \frac{mn}{n^2 - m^2} &= 0 \end{aligned}\right\} \tag{8}$$

where $n = 1, 2, 3, \ldots$ and $m \pm n$ is odd.

Equations (8) have a solution in which the coefficients a_n and the coefficients b_n are not all zero (that is, the plate has buckled) only if the following determinant vanishes:

$$
\begin{array}{c|cccccccccccccc|}
 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
n=1 & \frac{1}{k_s}M_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} & \cdots \\
n=2 & 0 & \frac{1}{k_s}M_2 & 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 0 & \frac{6}{5} & 0 & \frac{10}{21} & 0 & \cdots \\
n=3 & 0 & 0 & \frac{1}{k_s}M_3 & 0 & 0 & 0 & \cdots & 0 & -\frac{6}{5} & 0 & \frac{12}{7} & 0 & \frac{2}{3} & \cdots \\
n=4 & 0 & 0 & 0 & \frac{1}{k_s}M_4 & 0 & 0 & \cdots & -\frac{4}{15} & 0 & -\frac{12}{7} & 0 & \frac{20}{9} & 0 & \cdots \\
n=5 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_5 & 0 & \cdots & 0 & -\frac{10}{21} & 0 & -\frac{20}{9} & 0 & \frac{30}{11} & \cdots \\
n=6 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_6 & \cdots & -\frac{6}{35} & 0 & -\frac{2}{3} & 0 & -\frac{30}{11} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
n=1 & 0 & -\frac{2}{3} & 0 & -\frac{4}{15} & 0 & -\frac{6}{35} & \cdots & \frac{1}{k_s}M_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
n=2 & \frac{2}{3} & 0 & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 & \cdots & 0 & \frac{1}{k_s}M_2 & 0 & 0 & 0 & 0 & \cdots \\
n=3 & 0 & \frac{6}{5} & 0 & -\frac{12}{7} & 0 & -\frac{2}{3} & \cdots & 0 & 0 & \frac{1}{k_s}M_3 & 0 & 0 & 0 & \cdots \\
n=4 & \frac{4}{15} & 0 & \frac{12}{7} & 0 & -\frac{20}{9} & 0 & \cdots & 0 & 0 & 0 & \frac{1}{k_s}M_4 & 0 & 0 & \cdots \\
n=5 & 0 & \frac{10}{21} & 0 & \frac{20}{9} & 0 & -\frac{30}{11} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_5 & 0 & \cdots \\
n=6 & \frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\qquad (9)
$$

where

$$M_n = \frac{\pi}{8\beta}\left[\left(n^2 + \beta^2\right)^2 + \frac{12Z^2\beta^4}{\pi^4\left(n^2 + \beta^2\right)^2}\right]$$

By a rearrangement of rows and columns the infinite determinant can be factored into two infinite subdeterminants which have the same expansion and are therefore equivalent.

$$\begin{array}{c|ccccccccccccccc|l}
 & a_1 & b_2 & a_3 & b_4 & a_5 & b_6 & \cdots & b_1 & a_2 & b_3 & a_4 & b_5 & a_6 & \cdots & \\
n=1 & \frac{1}{k_s}M_1 & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=2 & \frac{2}{3} & \frac{1}{k_s}M_2 & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=3 & 0 & -\frac{6}{5} & \frac{1}{k_s}M_3 & \frac{12}{7} & 0 & \frac{2}{3} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=4 & \frac{4}{15} & 0 & \frac{12}{7} & \frac{1}{k_s}M_4 & -\frac{20}{9} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=5 & 0 & -\frac{10}{21} & 0 & -\frac{20}{9} & \frac{1}{k_s}M_5 & \frac{30}{11} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=6 & \frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & \frac{1}{k_s}M_6 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & = 0 \\
n=1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{k_s}M_1 & -\frac{2}{3} & 0 & -\frac{4}{15} & 0 & -\frac{6}{35} & \cdots & \\
n=2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & \frac{1}{k_s}M_2 & \frac{6}{5} & 0 & \frac{10}{21} & 0 & \cdots & \\
n=3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{6}{5} & \frac{1}{k_s}M_3 & -\frac{12}{7} & 0 & -\frac{2}{3} & \cdots & \\
n=4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{4}{15} & 0 & -\frac{12}{7} & \frac{1}{k_s}M_4 & \frac{20}{9} & 0 & \cdots & \\
n=5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{10}{21} & 0 & \frac{20}{9} & \frac{1}{k_s}M_5 & -\frac{30}{11} & \cdots & \\
n=6 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{6}{35} & 0 & -\frac{2}{3} & 0 & -\frac{30}{11} & \frac{1}{k_s}M_6 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & &
\end{array} \qquad (10)$$

The expansion of equation (10) may be approximated by finite subdeterminants. The first approximate expansion of equation (10) which will yield a value for k_s is obtained from a second-order determinant taken from either factor. Thus,

$$k_s^2 = \left(\frac{3}{2}\right)^2 M_1 M_2 \tag{11}$$

The second approximation, obtained from a third-order determinant, is given by

$$k_s^2 = \frac{M_1 M_2 M_3}{\left(\frac{6}{5}\right)^2 M_1 + \left(\frac{2}{3}\right)^2 M_3} \tag{12}$$

The third approximation, obtained from a fourth-order determinant, is given by

$$k_s^4 \left(\frac{8}{7} + \frac{8}{25}\right)^2 - k_s^2 \left[\left(\frac{12}{7}\right)^2 M_1 M_2 + \left(\frac{6}{5}\right)^2 M_1 M_4 + \left(\frac{4}{15}\right)^2 M_2 M_3 + \left(\frac{2}{3}\right)^2 M_3 M_4\right] + M_1 M_2 M_3 M_4 = 0 \tag{13}$$

Each of these equations shows that for a selected value of the curvature parameter Z the critical stress of long curved plates depends upon the wave length. Since a structure buckles at the lowest stress at which instability can occur, k_s is minimized with respect to the wave length by substituting values of β into equations (11), (12), or (13) until the minimum value of k_s can be obtained from a plot of k_s against β. Table 1 shows the convergence of the various approximations for k_s. The results are also shown graphically in figure 1.

Solution for plates with clamped edges (case I).- A procedure similar to that for plates with simply supported edges is followed for plates with clamped edges. Solutions are given for two types of clamped-edge support corresponding to two sets of boundary conditions.

The deflection function for case I

$$w = \sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} a_m\left[\cos\frac{(m-1)\pi y}{b} - \cos\frac{(m+1)\pi y}{b}\right]$$

$$+ \cos\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} b_m\left[\cos\frac{(m-1)\pi y}{b} - \cos\frac{(m+1)\pi y}{b}\right] \qquad (14)$$

satisfies the conditions on radial displacement w and implies that at the edges the axial displacement u is unrestrained and the circumferential displacement v is zero (see reference 3). Comparison of equation (14) with equation (4) shows that

$$\left.\begin{aligned} V_n &= \sin\frac{\pi x}{\lambda}\left[\cos\frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right] \\ W_n &= \cos\frac{\pi x}{\lambda}\left[\cos\frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right] \end{aligned}\right\} \qquad (15)$$

When operations equivalent to those carried out for a plate with simply supported edges are performed, the following simultaneous equations result:

For $n = 1$

$$a_1\left(2M_0 + M_2\right) - a_3M_2 + k_s \sum_{m=2,4}^{\infty} b_m \left[\frac{(m + 1)^2}{(m + 1)^2 - 4} - \frac{(m - 1)^2}{(m - 1)^2 - 4}\right] = 0$$

For $n = 2$

$$a_2\left(M_1 + M_3\right) - a_4M_3 + k_s \sum_{m=1,3}^{\infty} b_m \left[\frac{(m - 1)^2}{(m - 1)^2 - 1} - \frac{(m - 1)^2}{(m - 1)^2 - 9} - \frac{(m + 1)^2}{(m + 1)^2 - 1} + \frac{(m + 1)^2}{(m + 1)^2 - 9}\right] = 0$$

For $n = 3,4,\ldots$

$$a_n\left(M_{n-1} + M_{n+1}\right) - a_{n-2}M_{n-1} - a_{n+2}M_{n+1}$$

$$+ k_s \sum_{m=1}^{\infty} b_m \left[\frac{(m - 1)^2}{(m - 1)^2 - (n - 1)^2} - \frac{(m - 1)^2}{(m - 1)^2 - (n + 1)^2} - \frac{(m + 1)^2}{(m + 1)^2 - (n - 1)^2} + \frac{(m + 1)^2}{(m + 1)^2 - (n + 1)^2}\right] = 0$$

where $m + n$ is odd

For $n = 1$

$$b_1\left(2M_0 + M_2\right) - b_3M_2 - k_s \sum_{m=2,4}^{\infty} a_m \left[\frac{(m + 1)^2}{(m + 1)^2 - 4} - \frac{(m - 1)^2}{(m - 1)^2 - 4}\right] = 0$$

For $n = 2$

$$b_2\left(M_1 + M_3\right) - b_4M_3 - k_s \sum_{m=1,3}^{\infty} a_m \left[\frac{(m - 1)^2}{(m - 1)^2 - 1} - \frac{(m - 1)^2}{(m - 1)^2 - 9} - \frac{(m + 1)^2}{(m + 1)^2 - 1} + \frac{(m + 1)^2}{(m + 1)^2 - 9}\right] = 0$$

For $n = 3,4,\ldots$

$$b_n\left(M_{n-1} + M_{n+1}\right) - b_{n-2}M_{n-1} - b_{n+2}M_{n+1}$$

$$- k_s \sum_{m=1}^{\infty} a_m \left[\frac{(m - 1)^2}{(m - 1)^2 - (n - 1)^2} - \frac{(m - 1)^2}{(m - 1)^2 - (n + 1)^2} - \frac{(m + 1)^2}{(m + 1)^2 - (n - 1)^2} + \frac{(m + 1)^2}{(m + 1)^2 - (n + 1)^2}\right] = 0$$

where $m + n$ is odd

(16)

and where

$$M_n = \frac{\pi}{8\beta}\left[\left(n^2 + \beta^2\right)^2 + \frac{12Z^2\beta^4}{\pi^4\left(n^2 + \beta^2\right)^2}\right]$$

The infinite determinant formed by equations (16) can be rearranged so as to factor into two mutually equivalent infinite subdeterminants as in the solution for long curved plates with simply supported edges. The vanishing of the first of these factors is expressed by the following equation:

$$\begin{array}{c|cccccccc|l}
 & a_1 & b_2 & a_3 & b_4 & a_5 & b_6 & \cdots & & \\
n=1 & \frac{1}{k_s}(2M_0+M_2) & \frac{32}{15} & -\frac{1}{k_s}M_2 & -\frac{64}{105} & 0 & -\frac{32}{315} & \cdots & & \\
n=2 & \frac{32}{15} & \frac{1}{k_s}(M_1+M_3) & -\frac{352}{105} & -\frac{1}{k_s}M_3 & \frac{32}{35} & 0 & \cdots & & \\
n=3 & -\frac{1}{k_s}M_2 & -\frac{352}{105} & \frac{1}{k_s}(M_2+M_4) & \frac{1472}{315} & -\frac{1}{k_s}M_4 & -\frac{1376}{1155} & \cdots & & \\
n=4 & -\frac{64}{105} & -\frac{1}{k_s}M_3 & \frac{1472}{315} & \frac{1}{k_s}(M_3+M_5) & -\frac{4160}{693} & -\frac{1}{k_s}M_5 & \cdots & & = 0 \\
n=5 & 0 & \frac{32}{35} & -\frac{1}{k_s}M_4 & -\frac{4160}{693} & \frac{1}{k_s}(M_4+M_6) & \frac{9440}{1287} & \cdots & & \\
n=6 & -\frac{32}{315} & 0 & -\frac{1376}{1155} & -\frac{1}{k_s}M_5 & \frac{9440}{1287} & \frac{1}{k_s}(M_5+M_7) & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array} \qquad (17)$$

The first approximation, obtained from the second-order determinant, is given by

$$k_s^2 = \left(\frac{15}{32}\right)^2\left(2M_0 + M_2\right)\left(M_1 + M_3\right) \qquad (18)$$

The second approximation, obtained from the third-order determinant, is given by

$$k_s^2 = \frac{\left(M_1 + M_3\right)\left[\left(2M_0 + M_2\right)\left(M_2 + M_4\right) - M_2^2\right]}{\left(\frac{32}{15}\right)^2\left(M_2 + M_4\right) - 2\left(\frac{32}{15}\right)\left(\frac{352}{105}\right)M_2 + \left(\frac{352}{105}\right)^2\left(2M_0 + M_2\right)} \tag{19}$$

The third approximation, obtained from the fourth-order determinant, is given by

$$\begin{aligned}
&k_s^4\left[\left(\frac{32}{15}\right)\left(\frac{1472}{315}\right) - \left(\frac{352}{105}\right)\left(\frac{64}{105}\right)\right]^2 - k_s^2\left\{\left(\frac{1472}{315}\right)^2\left(2M_0 + M_2\right)\left(M_1 + M_3\right)\right. \\
&\quad + \left(\frac{352}{105}\right)^2\left(2M_0 + M_2\right)\left(M_3 + M_5\right) + \left(\frac{64}{105}\right)^2\left(M_1 + M_3\right)\left(M_2 + M_4\right) \\
&\quad + \left(\frac{32}{15}\right)^2\left(M_2 + M_4\right)\left(M_3 + M_5\right) - 2\left(\frac{64}{105}\right)\left(\frac{1472}{315}\right)M_2\left(M_1 + M_3\right) \\
&\quad - 2\left(\frac{32}{15}\right)\left(\frac{352}{105}\right)M_2\left(M_3 + M_5\right) - 2\left(\frac{352}{105}\right)\left(\frac{1472}{315}\right)M_3\left(2M_0 + M_2\right) \\
&\quad \left. - 2\left(\frac{32}{15}\right)\left(\frac{64}{105}\right)M_3\left(M_2 + M_4\right) + 2\left[\left(\frac{64}{105}\right)\left(\frac{352}{105}\right) + \left(\frac{32}{15}\right)\left(\frac{1472}{315}\right)\right]M_2M_3\right\} \\
&\quad + \left[2M_0\left(M_2 + M_4\right) + M_2M_4\right]\left[M_1\left(M_3 + M_5\right) + M_3M_5\right] = 0
\end{aligned} \tag{20}$$

These equations were used for values of Z between 0 and 30. As Z increases, the first term of the determinant becomes unimportant, and for $Z = 10^2$ equations similar to equations (18) to (20) and containing the b_2, a_3, b_4, and a_5 terms were used. For $Z = 10^3$ and $Z = 10^5$ the second term of the determinant also becomes unimportant, and equations similar to equations (18) to (20) but including the a_3, b_4, a_5, and b_6 terms were used. Each of these equations may be solved in the same manner as in the problem of curved plates with simply supported edges - that is, by substituting values of β into the equation for each value of Z until the minimum value of k_s is obtained from a plot of β against the corresponding values of k_s. Table 2 shows the convergence of the various approximations for k_s. The results are shown graphically in figure 1.

Solution for plates with clamped edges (case II).- Another deflection function for a plate with clamped edges (case II) is

$$w = \sin\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} a_m\left[\frac{1}{m}\sin\frac{m\pi y}{b} - \frac{1}{m+2}\sin\frac{(m+2)\pi y}{b}\right] + \cos\frac{\pi x}{\lambda}\sum_{m=1}^{\infty} b_m\left[\frac{1}{m}\sin\frac{m\pi y}{b} - \frac{1}{m+2}\sin\frac{(m+2)\pi y}{b}\right] \qquad (21)$$

This series satisfies the conditions on w and in addition implies that u is 0 and v is unrestrained at the edges (see reference 3). Comparison of equation (21) with equation (4) shows that in case II

$$\left.\begin{aligned} V_n &= \sin\frac{\pi x}{\lambda}\left[\frac{1}{n}\sin\frac{n\pi y}{b} - \frac{1}{n+2}\sin\frac{(n+2)\pi y}{b}\right] \\ W_n &= \cos\frac{\pi x}{\lambda}\left[\frac{1}{n}\sin\frac{n\pi y}{b} - \frac{1}{n+2}\sin\frac{(n+2)\pi y}{b}\right] \end{aligned}\right\} \qquad (22)$$

where $n = 1,2,3, \ldots$

When operations equivalent to those carried out for the case of simply supported edges are performed, the following simultaneous equations result:

For $n = 1$

$$a_1\left(M_1 + \frac{M_3}{9}\right) - \frac{a_3}{9} M_3 - k_s \sum_{m=2,4}^{\infty} b_m \left[\frac{1}{1 - m^2} - \frac{1}{1 - (m+2)^2} - \frac{1}{9 - m^2} + \frac{1}{9 - (m+2)^2}\right] = 0$$

For $n = 2$

$$a_2\left(\frac{M_2}{4} + \frac{M_4}{16}\right) - \frac{a_4}{16} M_4 - k_s \sum_{m=1,3}^{\infty} b_m \left[\frac{1}{4 - m^2} - \frac{1}{4 - (m+2)^2} - \frac{1}{16 - m^2} + \frac{1}{16 - (m+2)^2}\right] = 0$$

For $n = 3, 4, \ldots$

$$a_n\left(\frac{M_n}{n^2} + \frac{M_{n+2}}{(n+2)^2}\right) - \frac{a_{n-2}}{n^2} M_n - \frac{a_{n+2}}{(n+2)^2} M_{n+2}$$

$$- k_s \sum_{m=1}^{\infty} b_m \left[\frac{1}{n^2 - m^2} - \frac{1}{n^2 - (m+2)^2} - \frac{1}{(n+2)^2 - m^2} + \frac{1}{(n+2)^2 - (n+2)^2}\right] = 0$$

where $m + n$ is odd

For $n = 1$

$$b_1\left(M_1 + \frac{M_3}{9}\right) - \frac{b_3}{9} M_3 + k_s \sum_{m=2,4}^{\infty} a_m \left[\frac{1}{1 - m^2} - \frac{1}{1 - (m+2)^2} - \frac{1}{9 - m^2} + \frac{1}{9 - (m+2)^2}\right] = 0$$

For $n = 2$

$$b_2\left(\frac{M_2}{4} + \frac{M_4}{16}\right) - \frac{b_4}{16} M_4 + k_s \sum_{m=1,3}^{\infty} a_m \left[\frac{1}{4 - m^2} - \frac{1}{4 - (m+2)^2} - \frac{1}{16 - m^2} + \frac{1}{16 - (m+2)^2}\right] = 0$$

For $n = 3, 4, \ldots$

$$b_n\left(\frac{M_n}{n^2} + \frac{M_{n+2}}{(n+2)^2}\right) - \frac{b_{n-2}}{n^2} M_n - \frac{b_{n+2}}{(n+2)^2} M_{n+2}$$

$$+ k_s \sum_{m=1}^{\infty} a_m \left[\frac{1}{n^2 - m^2} - \frac{1}{n^2 - (m+2)^2} - \frac{1}{(n+2)^2 - m^2} + \frac{1}{(n+2)^2 - (m+2)^2}\right] = 0$$

where $m + n$ is odd

(23)

and where

$$M_n = \frac{\pi}{8\beta}\left[\left(n^2 + \beta^2\right)^2 + \frac{12Z^2\beta^4}{\pi^4\left(n^2 + \beta^2\right)^2}\right]$$

The infinite determinant formed by equations (23) can be rearranged so as to factor into two mutually equivalent infinite subdeterminants. The vanishing of either of these factors leads to a relation between k_s, β, and Z. From this relation the value of k_s is found for a given value of Z by minimizing k_s with respect to β. Because the solution is slowly convergent, tenth-order determinants were used. These determinants were evaluated by the Crout method (reference 5) for assumed values of k_s and β. For given values of β, corresponding values of k_s which caused the determinants to vanish were found. The critical value of k_s is the minimum value of k_s found from a plot of k_s against the corresponding values of β. Table 2 presents the theoretical values of k_s obtained by solving determinants of different order. The results are shown graphically by the dashed curve in figure 1.

REFERENCES

1. Leggett, D. M. A.: The Elastic Stability of a Long and Slightly Bent Rectangular Plate under Uniform Shear. Proc. Roy. Soc. (London), ser. A., vol. 162, no. 908, Sept. 1, 1937, pp. 62-83.

2. Kromm, A.: The Limit of Stability of a Curved Plate Strip under Shear and Axial Stresses. NACA TM No. 898, 1939.

3. Batdorf, S. B.: A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells. II - Modified Equilibrium Equation. NACA TN No. 1342, 1947.

4. Duncan, W. J.: The Principles of the Galerkin Method. R. & M. No. 1848, British A.R.C., 1938.

5. Crout, Prescott D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. Supp. to Elec. Eng., Trans. Section, AIEE, vol. 60, Dec. 1941, pp. 1235-1240. (Abridged as Marchant Methods MM-182, Sept. 1941, Marchant Calculating Machine Co., Oakland, Calif.)

TABLE 1

THEORETICAL SHEAR-STRESS COEFFICIENTS AND RATIO OF WIDTH OF PLATE TO HALF WAVE LENGTH OF BUCKLE FOR LONG PLATES WITH TRANSVERSE CURVATURE WITH SIMPLY SUPPORTED EDGES

Z	First approximation		Second approximation		Third approximation	
	k_s	β	k_s	β	k_s	β
0	5.60	0.77	5.34	0.80	------	-----
1	5.65	.70	5.38	.80	------	-----
2	5.68	.78	5.42	.80	------	-----
5	6.04	.72	5.76	.76	------	-----
10	7.06	.58	6.72	.65	------	-----
30	11.16	.33	10.66	.35	------	-----
100	20.02	.17	19.20	.18	------	-----
300	34.3	.10	33.0	.098	------	-----
1,000	62.60	.054	60.15	.055	60.02	0.055
10,000	198.5	.017	190.0	.018	189.7	.018
100,000	626.0	.005	601.2	.006	------	-----

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 2

THEORETICAL SHEAR-STRESS COEFFICIENTS AND RATIO OF WIDTH OF PLATE TO HALF WAVE LENGTH OF BUCKLE FOR LONG PLATES WITH TRANSVERSE CURVATURE WITH CLAMPED EDGES

Z	First approximation		Second approximation		Third approximation	
	k_s	β	k_s	β	k_s	β
Case I (u, unrestrained; v = 0 at edges)						
0	9.55	1.18	9.31	1.21	9.09	1.21
1	9.59	1.19	9.34	1.23	-------	-----
2	9.70	1.19	9.43	1.23	-------	-----
5	10.46	1.25	10.00	1.29	-------	-----
10	12.69	1.38	11.49	1.41	-------	-----
30	28.23	1.82	18.44	1.62	18.10	1.66
100	44.42	.22	34.3	1.36	-------	-----
1,000	125.8	.41	110.7	.56	110.7	.56
100,000	1280	.013	1112.8	.017	1112.6	.017
Z	Fourth-order determinant		Eighth-order determinant		Tenth-order determinant	
	k_s	β	k_s	β	k_s	β
Case II (u = 0; v, unrestrained at edges)						
0	10.44	1.29	9.82	1.28	9.66	1.25
30	------	-----	16.99	1.31	------	-----
100	33.53	1.28	30.43	1.08	------	-----
10,000	336.6	.126	308.2	.118	303.3	.114

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

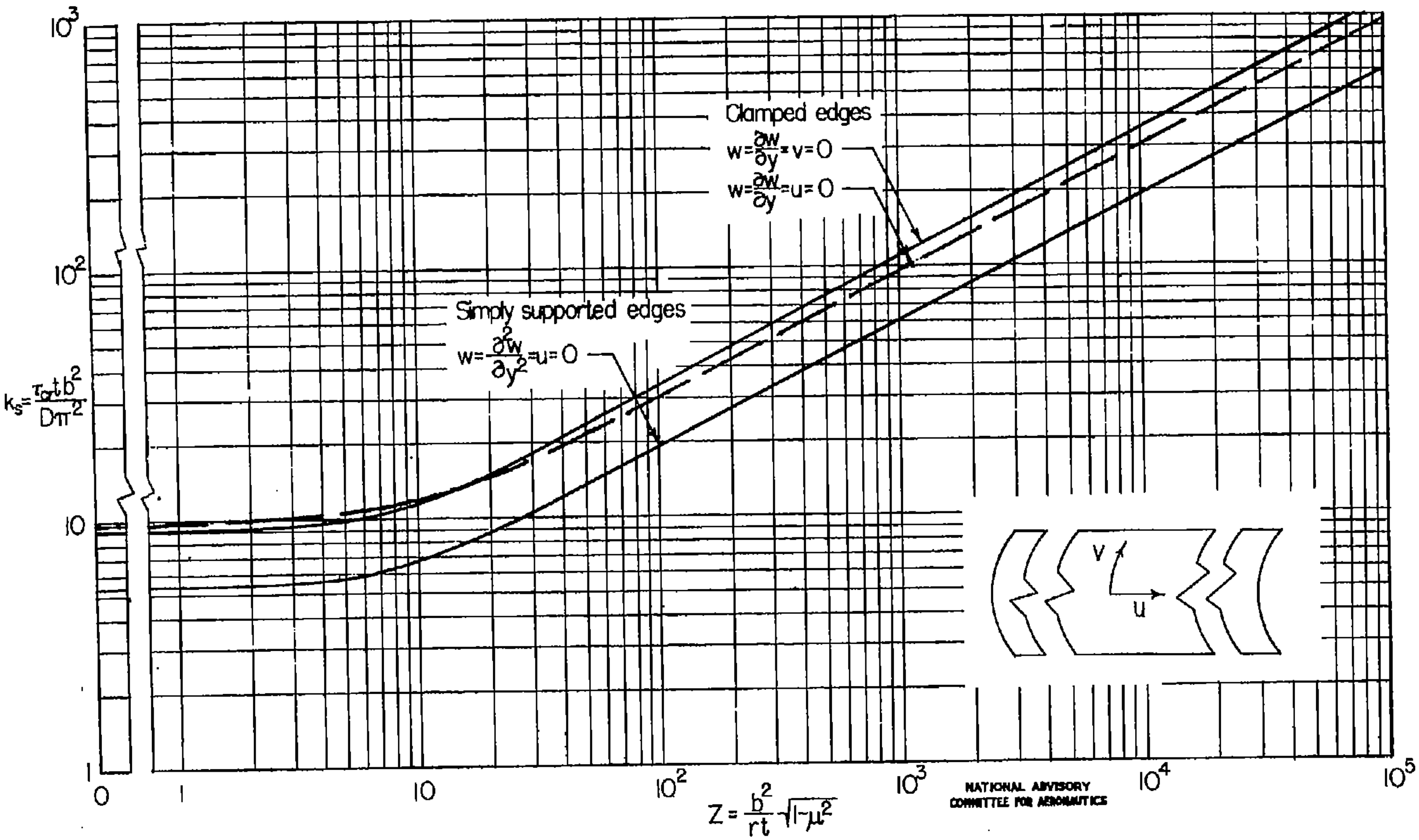

Figure 1.- Critical-shear-stress coefficients for long plates with transverse curvature having simply supported or clamped edges.

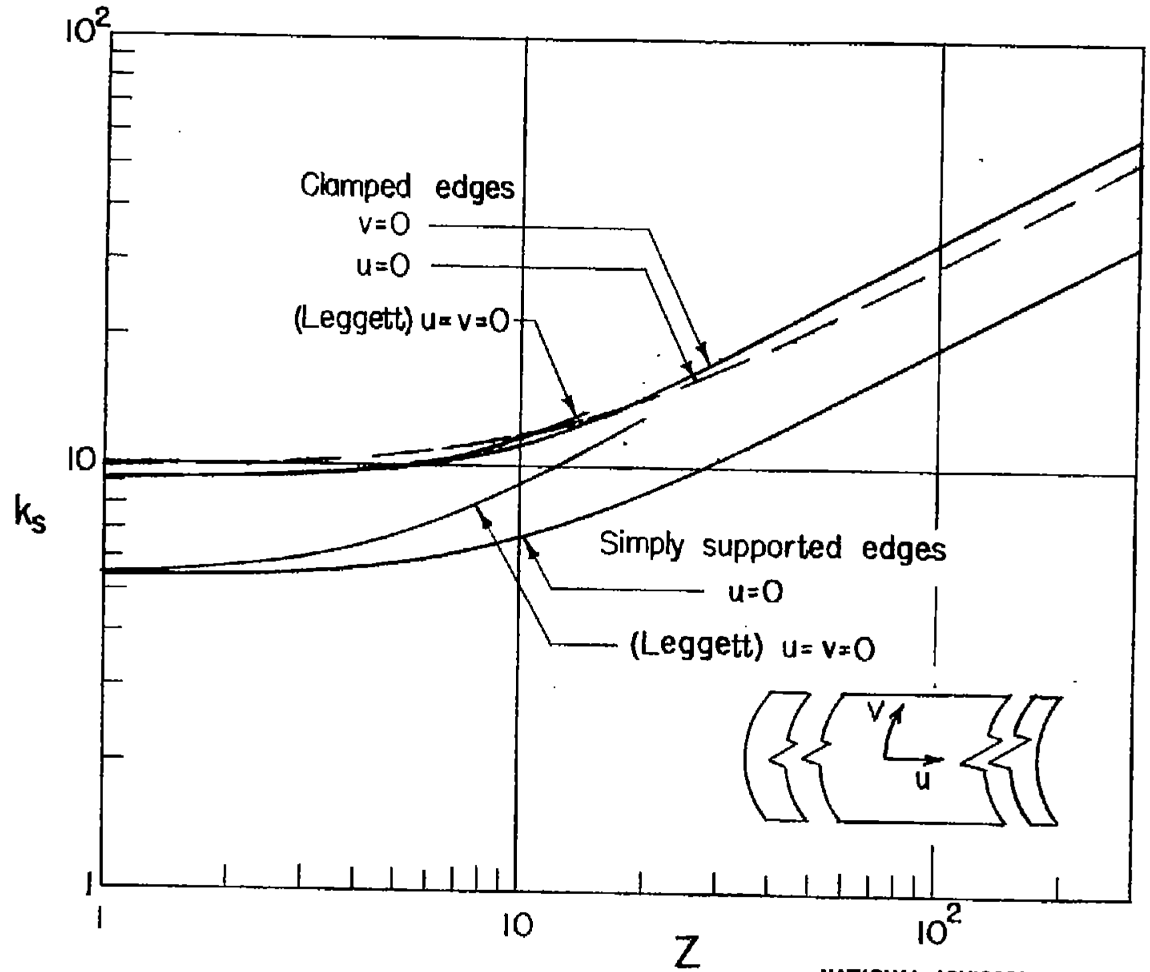

Figure 2.- Comparison of present solution with Leggett's results.